Optimal Edge Deletions for Signed Graph Balancing

Falk Hüffner Nadja Betzler Rolf Niedermeier

Friedrich-Schiller-Universität Jena
Institut für Informatik

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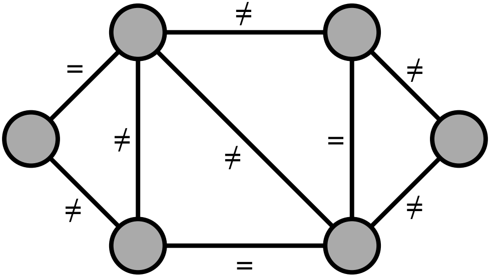
Outline

Balanced graphs

Definition

A graph with edges labeled by $=$ or $\neq$ (signed graph) is balanced if the vertices can be colored with two colors such that the relation on each edge holds.



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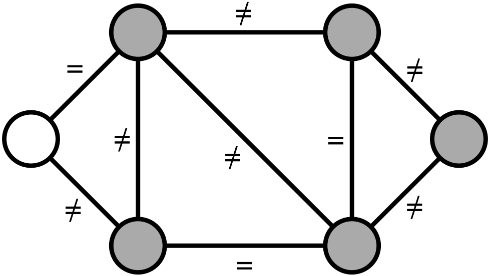

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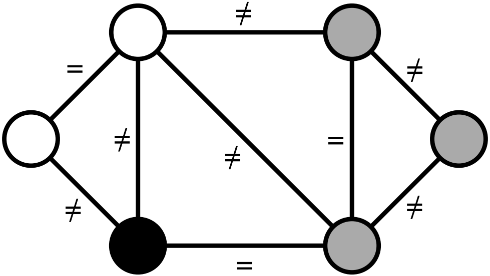

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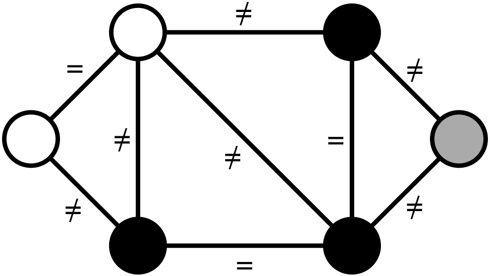

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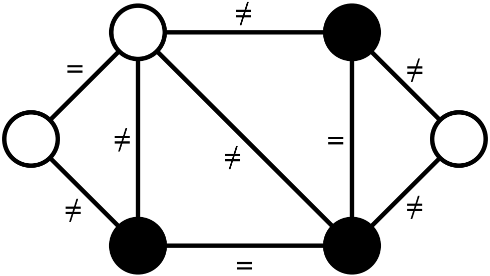

Characterization of balance

Special case

Bipartite graphs are balanced graphs that contain only $\neq$-edges.

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Theorem (Kőnig 1936)

A signed graph is balanced iff it contains no cycle with an odd number of $\neq$-edges.

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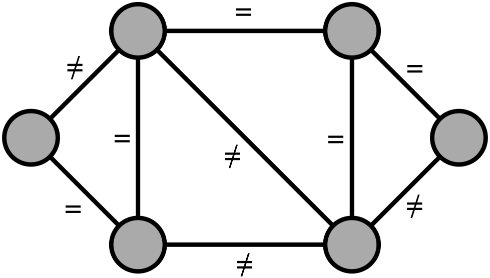
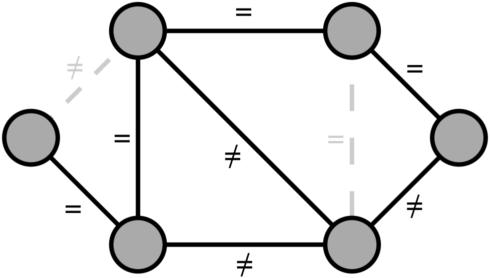
Corollary

Bipartite graphs are graphs that contain no cycle of odd length.

Balanced Subgraph



Balanced Subgraph



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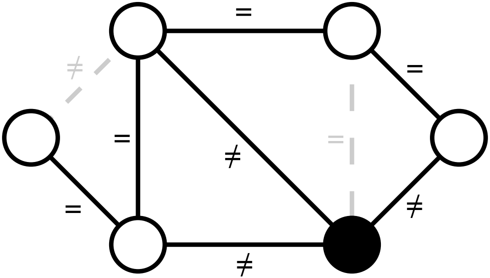

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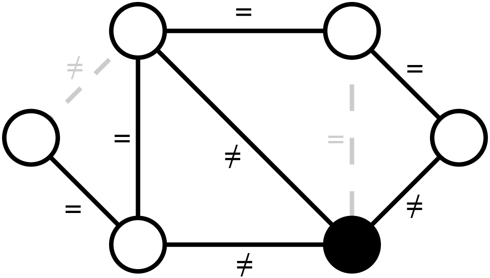


Definition (Balanced Subgraph)

Input: A graph with edges labeled by $=$ or $\neq$.
Task: Find a minimum set of edges to delete such that the graph becomes balanced.

Applications of Balanced Subgraph

- "Monotone subsystems" in biological networks
  [DasGupta et al., WEA 2006]
- Balance in social networks
  [Harary, Mich. Math. J. 1953]
- Portfolio risk analysis
  [Harary et al., IMA J. Manag. Math. 2002]
- Minimum energy state of magnetic materials (spin glasses)
  [Kasteleyn, J. Math. Phys. 1963]
- Stability of fullerenes
  [Došlić&Vikičević, Discr. Appl. Math. 2007]
- Integrated circuit design
  [Chiang et al., IEEE Trans. CAD of IC&Sys. 2007]

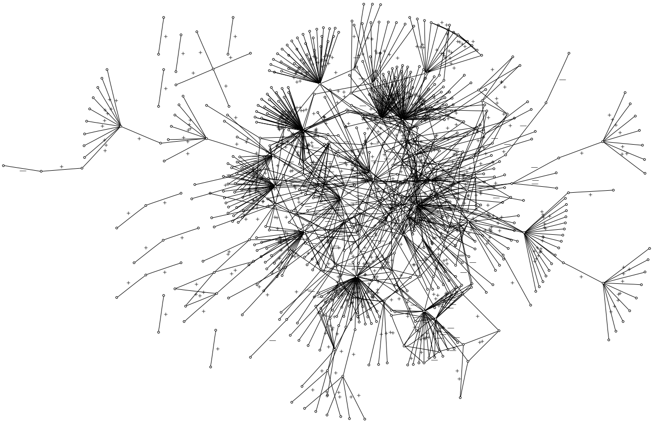
Balanced Subgraph: known results

- BALANCED SUBGRAPH is NP-hard, since it is a generalization of MAX-CUT (MAX-CUT is the special case where all edges are $\neq$)
- A solution that keeps at least 87.8 % of the edges can be found in polynomial time [DASGUPTA et al., WEA 2006]
- A solution that deletes at most c times the edges that need to be deleted can probably not be found in polynomial time [KHOT, STOC 2002]

Graph structure

Idea

Exploit the structure of the relevant networks



Data reduction

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Replace the instance by a simpler, equivalent one.

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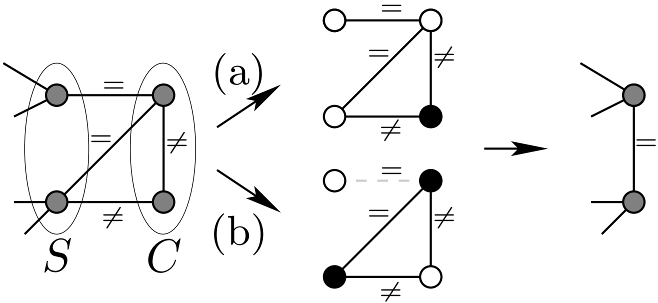
Example

Delete all degree-1 vertices.

Separator-based data reduction



Data reduction scheme

Data reduction scheme

- Find separator S that cuts off small component C
- For each of the (up to symmetry) $2^{|S|-1}$ colorings of S, determine the size of an optimal solution for $G[S \cup C]$
- Replace in G the subgraph $G[S \cup C]$ by an equivalent smaller gadget

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Subsumes all 8 data reduction rules given by [Wernicke, 2003] for Edge Bipartization

Filling in the data reduction scheme

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  - Separators of size 0 and 1 can be found in linear time by depth-first search [Gabow, IPL 2000]
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- How to construct gadgets that behave equivalently to $S \cup C$?

Gadget construction

Idea

Use atomic gadgets and describe their effect by cost vectors.

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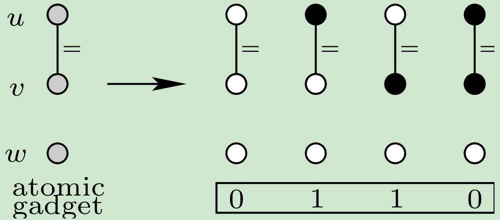

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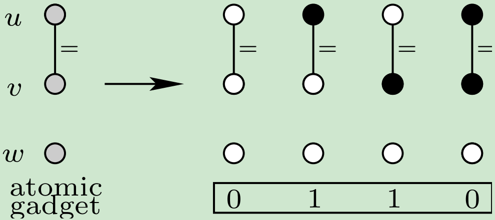

Theorem

With 10 atomic gadgets, we can emulate the behavior of any component behind a 3-vertex cut.

Gadget construction

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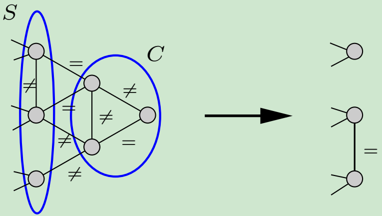

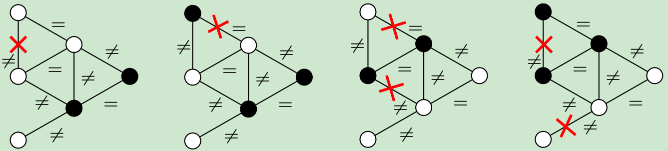

1	1	2	2	−(1,1,1,1)
0	0	1	1	−(0,0,1,1)
0	0	0	0	

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- In our implementation: simple branch & bound
- Sometimes this is a bottleneck!

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All separators with $|S| = 2$ and $|C| \geq 1$ and and all separators with $|S| = 3$ and $|C| \geq 2$ are subject to data reduction.

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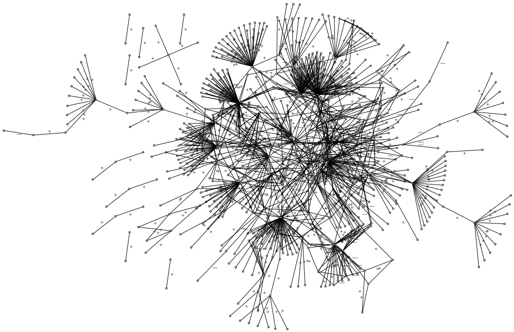
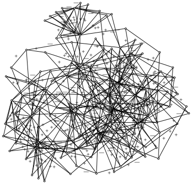
- 4-cuts: 2948 atomic gadgets (heuristically found)

Reduction. . . and then?



$n = 690,\ m = 1082$



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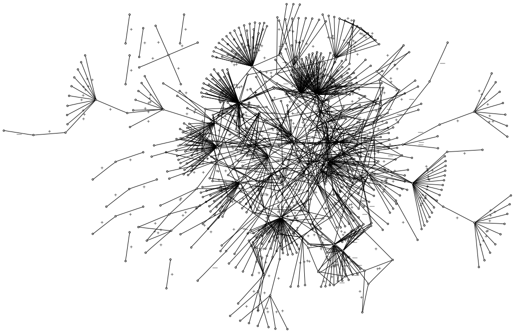

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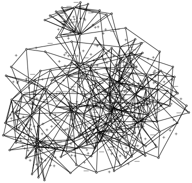

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After data reduction, a hard "core" remains.

Fixed-parameter tractability

Idea

Exploit the fact that biological networks are close to being balanced (i. e., the number k of edges that need to be deleted to make them balanced is small).

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BALANCED SUBGRAPH *can be solved in* $O(2^k \cdot m^2)$ *time by a reduction to* EDGE BIPARTIZATION *and using an algorithm based on iterative compression* [Guo et al., JCSS 2006].

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A heuristic speedup trick can give large speedups over this worst-case running time.

Experimental results

			Approximation			Exact alg.	
Data set	n	m	$k \geq$	$k \leq$	t [min]	k	t [min]
EGFR	330	855	196	219	7	210	108
Yeast	690	1082	0	43	77	41	1
Macr.	678	1582	218	383	44	374	1

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- Yeast is not solvable without reducing 4-cuts
- A real-world network with 688 vertices and 2208 edges could not be solved

Outlook

- Directed case of Balanced Subgraph
  - Problem: Characterization by two-coloring holds only for strongly connected graphs
- The data reduction scheme is applicable to all graph problems where a coloring or a subset of the vertices is sought. For example:
  - Vertex Cover
  - Dominating Set
  - 3-Coloring
  - Feedback Vertex Set